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An explicit criterion to determine the number of roots in an interval of a polynomial *

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Abstract The negative root discriminant sequence (n.r.d.) for a given polynomial with general symbolic coefficients is a set of explicit expressions in terms of the coefficients that are sufficient for determining the number of distinct negative/positive roots and thus can be used to determine the number of roots in an interval of the given polynomial. Some interesting properties related to n.r.d. are studied.

Keywords: **real root, polynomial, discriminant sequence, negative root discriminant sequence.**

Given a polynomial with real coefficients,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

we consider how to determine the number of roots in an interval (a, b) from the viewpoint of explicit criterion. Theoretically speaking, we can let

$$g(x) = (x^2+1)^n \cdot f\left(\frac{ax^2+b}{x^2+1}\right),$$

and find the number of all real roots of g (using the algorithm introduced in refs. [1, 2], which is also partly described in sec. 1 of this paper), then, half the number is what we want. But such a procedure is very inefficient indeed for symbolic computation, so we employ the following method in practice.

(i) Let $f_{(a,\infty)}$, $f_{(b,\infty)}$ and $f_{(a,b)}$ denote the numbers of roots of $f(x)$ in (a,∞), (b,∞), and (a,b), respectivly. Obviously $f_{(a,b)} = f_{(a,\infty)} - f_{(b,\infty)}$.

(ii) The problem of finding numbers $f_{(a,\infty)}$ and $f_{(b,\infty)}$ can be reduced (by a translation) to determining the number of positive roots of a polynomial.

(iii) A more efficient algorithm, an explicit criterion, to determine the number of positive/negative roots is described in section 2.

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The determination of the number of distinct roots in an interval can also be accomplished by using Sturm-Habicht sequence[3,4], but here only the principal minor sequence of the discrimination matrix (see sec. 1 for the definition) of the polynomial with a change of variable is employed.

1 Discriminant sequence for polynomials

Given a polynomial with general symbolic coefficients,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

the following $(2n+1) \times (2n+1)$ matrix in terms of the coefficients

$$\begin{bmatrix}
a_0 & a_1 & a_2 & \cdots & a_n & & & & \\
0 & na_0 & (n-1)a_1 & \cdots & a_{n-1} & & & & \\
 & a_0 & a_1 & \cdots & a_{n-1} & a_n & & & \\
 & 0 & na_0 & \cdots & 2a_{n-2} & a_{n-1} & & & \\
 & & & \cdots & \cdots & & & & \\
 & & & \cdots & \cdots & & & & \\
 & & & a_0 & a_1 & \cdots & \cdots & a_n & \\
 & & & 0 & na_0 & \cdots & \cdots & a_{n-1} & \\
 & & & & a_0 & a_1 & \cdots & \cdots & a_n
\end{bmatrix}$$

is called the discrimination matrix of $f(x)$, and denoted by $\mathrm{Discr}(f)$.

Let d_k or $d_k(f)$ denote the determinant of the submatrix of $\mathrm{Discr}(f)$, which is formed by the first k rows and the first k columns, for $k = 1, \cdots, 2n+1$. Let $D_k = d_{2k}$ for $k = 1, \cdots, n$. We call the n-tuple,

$$\{D_1, D_2, \cdots, D_n\}$$

the discriminant sequence of $f(x)$.

We call list

$$[\mathrm{sign}\ (B_1), \mathrm{sign}\ (B_2), \cdots, \mathrm{sign}\ (B_n)]$$

the sign list of a given sequence B_1, B_2, $\cdots$, B_n, where

$$\mathrm{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

Given a sign list $[s_1, s_2, \cdots, s_n]$, we construct a new list $[t_1, t_2, \cdots, t_n]$ which is called the revised sign list according to the following law.

(i) If $[s_i, s_{i+1}, \cdots, s_{i+j}]$ is a section of the given list, where

$$s_i \neq 0,\ s_{i+1} = \cdots = s_{i+j-1} = 0,\ s_{i+j} \neq 0,$$

then, we replace the subsection

$$[s_{i+1}, \cdots, s_{i+j-1}]$$

by the first $j-1$ terms of $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, \cdots]$, i.e. let

$$t_{i+r} = (-1)^{[(r+1)/2]} \cdot s_i,\ r = 1, 2, \cdots, j-1.$$

(ii) Otherwise, let $t_k = s_k$, i.e. no change in other terms.

Theorem 1 (refs. [1, 2]). *Given a polynomial $f(x)$ with real coefficients,*

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

if the number of sign changes of the revised sign list of

$$\{D_1(f), D_2(f), \cdots, D_n(f)\}$$

is ν, then the number of distinct pairs of conjugate imaginary roots of $f(x)$ equals ν. Furthermore, if the number of non-vanishing members of the revised sign list is l, then the number of distinct real roots of $f(x)$ equals $l - 2\nu$.

2 Negative root discriminant sequence for polynomials

Given a polynomial with real symbolic coefficients,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n,$$

denote the number of roots of $f(x)$ in (a, b) by $f_{(a,b)}$, let

$$\tilde{h}(x) = f(x^2),\ h(x) = f(-x^2),$$

and assume $f(0) \neq 0$. Then we have

$$f_{(0,\infty)} = \frac{1}{2}\tilde{h}_{(-\infty,\infty)},\ f_{(-\infty,0)} = \frac{1}{2}h_{(-\infty,\infty)}.$$

Let $\{d_1, d_2, \cdots, d_{2n+1}\}$ be the principal minor sequence of $\mathrm{Discr}(f)$, the discrimination matrix of $f(x)$, we call the $2n$-tuple

$$\{d_1 d_2,\ d_2 d_3,\ \cdots,\ d_{2n} d_{2n+1}\}$$

the negative root discriminant sequence of $f(x)$, and denote it by n.r.d.(f).

Theorem 2. *Let* $\{d_1, d_2, \cdots, d_{2n+1}\}$ *be the principal minor sequence of* Discr (f), *the discrimination matrix of the polynomial*,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and let $h(x) = f(-x^2)$. *Assume* $a_0 \neq 0$. *Then, the discriminant sequence of* $h(x)$,

$$\{D_1(h), D_2(h), \cdots, D_{2n}(h)\},$$

is equal to $n.r.d.(f)$,

$$\{d_1d_2, d_2d_3, \cdots, d_{2n}d_{2n+1}\},$$

i.e. $D_k(h) = d_k(f)d_{k+1}(f)$, *up to a factor of the same sign as* a_0, *for* $k = 1, 2, \cdots, 2n$.

Proof. (i) If k is even, say, $k = 2j, (1 \leqslant j \leqslant n)$, then,

$$D_k(h) = \begin{vmatrix} (-1)^n a_0 & 0 & (-1)^{n-1}a_1 & 0 & \cdots & 0 \\ 0 & (-1)^n 2na_0 & 0 & (-1)^{n-1}2(n-1)a_1 & \cdots & (-1)^{n-2j+1}2(n-2j+1)a_{2j-1} \\ & (-1)^n a_0 & 0 & (-1)^{n-1}a_1 & \cdots & (-1)^{n-2j+1}a_{2j-1} \\ & & (-1)^n 2na_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & (-1)^n a_0 & \cdots & \cdots & (-1)^{n-j}a_j \\ 0 & \cdots & 0 & (-1)^n 2na_0 & \cdots & 0 \end{vmatrix}_{4j\times 4j}$$

$$= (-1)^n 2^k a_0 \times$$

$$\begin{vmatrix} (-1)^n na_0 & 0 & (-1)^{n-1}(n-1)a_1 & 0 & \cdots & (-1)^{n-2j+1}(n-2j+1)a_{2j-1} \\ (-1)^n a_0 & 0 & (-1)^{n-1}a_1 & 0 & \cdots & (-1)^{n-2j+1}a_{2j-1} \\ 0 & (-1)^n na_0 & 0 & (-1)^{n-1}(n-1)a_1 & \cdots & 0 \\ 0 & (-1)^n a_0 & 0 & (-1)^{n-1}a_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \cdots & (-1)^n a_0 & \cdots & \cdots & (-1)^{n-j}a_j \\ \cdots & \cdots & 0 & (-1)^n na_0 & \cdots & 0 \end{vmatrix}.$$

Now, in the last determinant, we move in order the 2nd, 4th, 6th, $\cdots$and $(4j-2)$th columns to the first $(2j-1)$ columns, and then, move in order the 3rd, 4th, 7th, 8th, $\cdots$, $(4j-5)$th, $(4j-4)$th and$(4j-1)$th rows to the first $(2j-1)$ rows. We have

$$D_k(h) = (-1)^{\delta}\cdot(-1)^{n}\cdot 2^{k}\cdot a_0\cdot\begin{vmatrix} A & 0 \\ 0 & B\end{vmatrix}$$

where

$$\begin{aligned}\delta &= (2-1)+(4-2)+(6-3)+\cdots+(4j-2-2j+1)+\\ &\quad (3-1)+(4-2)+(7-3)+(8-4)+\cdots+(4j-1-2j+1)\\ &\equiv 1+2+3+\cdots+(2j-1)\ (\mathrm{mod}\ 2)\\ &\equiv j(\mathrm{mod}2),\end{aligned}$$

$$A=\begin{vmatrix} (-1)^n na_0 & (-1)^{n-1}(n-1)a_1 & \cdots & \cdots & (-1)^{n-2j+2}(n-2j+2)a_{2j-2} \\ (-1)^n a_0 & (-1)^{n-1}a_1 & \cdots & \cdots & (-1)^{n-2j+2}a_{2j-2} \\ & (-1)^n na_0 & \cdots & \cdots & \vdots \\ & (-1)^n a_0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & (-1)^n na_0 & \cdots & (-1)^{n-j+1}(n-j+1)a_{j-1}\end{vmatrix}_{(2j-1)\times(2j-1)}$$

and

$$B=\begin{vmatrix} (-1)^n na_0 & \cdots & \cdots & (-1)^{n-2j+1}(n-2j+1)a_{2j-1} \\ (-1)^n a_0 & \cdots & \cdots & (-1)^{n-2j+1}a_{2j-1} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & (-1)^n a_0 & \cdots & (-1)^{n-j}a_j\end{vmatrix}_{2j\times 2j}$$

Now, if n is even, let -1 time the 1st, 3rd, 5th, 7th, $\cdots$ columns of A, B respectively; otherwise, if n is odd, let -1 time the 2nd, 4th, 6th, 8th, $\cdots$ columns of A, B respectively. After that, let -1 time the 1st, 2nd, 5th, 6th, 9th, 10th, $\cdots$ rows of A, B respectively. Then, we get

$$A=(-1)^{2j}A^* = A^*,\quad B=(-1)^jB^*,\ \text{if } n\equiv 0\ (\mathrm{mod}\ 2),$$

$$A=(-1)^{2j-1}A^* = (-1)A^*,\ B=(-1)^jB^*,\ \text{if } n\equiv 1\ (\mathrm{mod}\ 2).$$

where

$$A^{*} = \begin{vmatrix} na_0 & (n-1)a_1 & \cdots & \cdots & (n-2j+2)a_{2j-2} \\ a_0 & a_1 & \cdots & \cdots & a_{2j-2} \\ & na_0 & \cdots & \cdots & \vdots \\ & a_0 & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ & & na_0 & \cdots & (n-j+1)a_{j-1} \end{vmatrix}_{(2j-1)\times(2j-1)},$$

$$B^{*} = \begin{vmatrix} na_0 & \cdots & \cdots & (n-2j+1)a_{2j-1} \\ a_0 & \cdots & \cdots & a_{2j-1} \\ \vdots & \vdots & \ddots & \vdots \\ \cdots & a_0 & \cdots & a_j \end{vmatrix}_{2j\times 2j}.$$

So, whatever n is, we have

$$\begin{aligned} D_k(h) &= (-1)^{\delta} \cdot (-1)^{n} \cdot 2^{k} \cdot a_0 \cdot A \cdot B \\ &= (-1)^{j} \cdot (-1)^{j} \cdot 2^{k} \cdot a_0 \cdot A^{*} \cdot B^{*} \\ &= 2^{k} \cdot a_0 \cdot A^{*} \cdot B^{*}. \end{aligned}$$

Noting

$$A^{*} = \frac{1}{a_0}\begin{vmatrix} a_0 & 0 \\ 0 & A^{*} \end{vmatrix} = \frac{1}{a_0} d_{2j}, \quad B^{*} = \frac{1}{a_0}\begin{vmatrix} a_0 & 0 \\ 0 & B^{*} \end{vmatrix} = \frac{1}{a_0} d_{2j+1},$$

we obtain

$$D_k(h) = \frac{2^{k}}{a_0} \cdot d_{2j} \cdot d_{2j+1}.$$

Remembering $k = 2j$, we have

$$D_k(h) = d_k(f) \cdot d_{k+1}(f),$$

up to a factor of the same sign as a_0.

(ii) Similarly, we can prove the case where k is odd.

Theorem 3. *Let* $\{d_1, d_2, \cdots, d_{2n+1}\}$ *be the principal minor sequence of* Discr(f), *the* *discrimination matrix of the polynomial*,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and let $\tilde{h}(x) = f(x^2)$. *Assume* $a_0 \neq 0$. *Ten, for each term of the discriminant sequence of* $\tilde{h}(x)$,

$$\{D_1(\tilde{h}),\ D_2(\tilde{h}),\ \cdots,\ D_{2n}(\tilde{h})\},$$

we have

$$D_k(\tilde{h}) = (-1)^{\left[\frac{k}{2}\right]} d_k(f) d_{k+1}(f),$$

up to a factor of the same sign as a_0.

Proof. Refer to the same discussion in Theorem 2 or see footnote 1) for details.

Theorem 4. *Let* $\{d_1, d_2, \cdots, d_{2n+1}\}$ *be the principal minor sequence of* Discr(f), *the discrimination matrix of polynomial* $f(x)$ *with* $a_0 \neq 0$, $a_n \neq 0$. *Denote the number of sign changes and the number of non-vanishing members of the revised sign list of sequence*,

$$\{d_1d_2,\ d_2d_3,\ \cdots,\ d_{2n}d_{2n+1}\},$$

by μ *and* $2l$, *respectively, then, the number of distinct negative roots of* $f(x)$ *equals* $l-\mu$.

Proof. It is the direct corollary of Theorems 1 and 2.

3 Some properties related to n.r.d.

Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n (a_0 \neq 0),$$

$$g(x) = b_0x^n + b_1x^{n-1} + \cdots + b_n,$$

and

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n & & & & \\ b_0 & b_1 & b_2 & \cdots & b_n & & & & \\ & a_0 & a_1 & a_2 & \cdots & a_n & & & \\ & b_0 & b_1 & b_2 & \cdots & b_n & & & \\ & & & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & & & & \\ & & & a_0 & a_1 & a_2 & \cdots & a_n \\ & & & b_0 & b_1 & b_2 & \cdots & b_n \end{bmatrix}_{2n\times 2n}$$

1) Yang, L., Xia, B. C., Explicit criterion to determine the number of positive roots of a polynomial, *MM Research Preprints*, 1997, 15: 134.

be the Sylvester matrix of $f(x)$ and $g(x)$. Let

$$\text{sturm}\ (f,g) = [r_0(x) = f(x), r_1(x) = g(x), r_2(x), \cdots]$$

be the standard Sturm sequence (ref. [5]) of $f(x)$ and $g(x)$, where

$$r_{k+1}(x) = -(r_{k-1}(x) - q_k(x)r_k(x)),\ \deg(r_{k+1}(x)) < \deg(r_k(x)),\ k = 1,2,\cdots.$$

Let

$$s_{-1} = 0,\ s_i = \deg(r_i(x)) - \deg(r_{i+1}(x)),\ i = 0,1,\cdots,$$

$$q_{-1} = 0,\ q_j = \sum_{i=-1}^{j-1} s_i,\ j = 0,1,\cdots,$$

$$\delta_k = \frac{1}{2}\sum_{p=-1}^{k-1}(s_p - 1)s_p,\ \overline{r_{-1}} = 1,\ \overline{r_i} = \text{lcoeff}(r_i(x)),\ i = 0,1,\cdots,$$

and A_k be the submatrix of A formed by the first $2k$ rows and the first $n+k$ columns, $A(k,l)$ the submatrix of A formed by the first k rows, first $k-1$ columns and the $(k+l)$th column.

Theorem 5.

(i) *If* $m \neq \sum_{i=0}^{k-1} s_i$, *then* $|A(2m,0)| = 0$;

(ii) *If* $m = \sum_{i=0}^{k-1} s_i$, *then*

$$|A(2m,0)| = (-1)^{\delta_k}\cdot(\overline{r_0 r_1})^{s_0}\cdot(\overline{r_1 r_2})^{s_1}\cdots(\overline{r_{k-1} r_k})^{s_{k-1}},$$

$$r_k(x) = \frac{\overline{r_k}}{|A(2m,0)|}\cdot\sum_{t=0}^{n-m}|A(2m,t)|\,x^{n-m-t}.$$

Proof. See refs. [1, 2].

Now, letting $H_0 = 1$ and $H_1, H_2, \cdots, H_n$ be the even order principal minor sequence of A, we have

Corollary 1.

(i) *If* $m \neq q_j = \sum_{i=0}^{j-1} s_i$ *for* $j = 1,\cdots,k$, *then* $H_m = 0$. *That is to say, in list* $\{H_0, H_1, \cdots, H_n\}$, *every member between* $H_{q_{j-1}}$ *and* H_{q_j} *is* 0.

(ii) *If* $m = q_j = \sum_{i=0}^{j-1} s_i$ *for a certain* j $(1 \leqslant j \leqslant k)$, *then*,

$$H_m = (-1)^{\delta_k}\cdot(\overline{r_0 r_1})^{s_0}\cdot(\overline{r_1 r_2})^{s_1}\cdots(\overline{r_{k-1} r_k})^{s_{k-1}}.$$

Let $\sigma_i = H_{q_i}$ *be the ith non-vanishing member in list* $\{H_1, \cdots, H_n\}$, *then*

$$\frac{\sigma_{i+1}}{\sigma_i} = (-1)^{(s_i-1)s_i/2}(\overline{r_i r_{i+1}})^{s_i}.$$

Proposition 1.

(i) *In list* $\{H_0, H_1, \cdots, H_n\}$, *if* $H_i = 0$ *and* $H_{i-1} \cdot H_{i+1} \neq 0$ *for some* i, $1 \leqslant i \leqslant n-1$, *then* $H_{i-1} \cdot H_{i+1} < 0$. *If*

$$H_{i-1} = H_i = H_{i+1} = 0, \quad H_{i-2} \cdot H_{i+2} \neq 0,$$

then $H_{i-2} \cdot H_{i+2} > 0$.

(ii) *Let* $h_1, h_2, \cdots, h_{2n-1}$, h_{2n} *be the principal minor sequence of* A, *hence* $H_i = h_{2i}$ *for* $i = 1, 2, \cdots, n$. *If* $h_{2m} = h_{2m+2} = 0$. *Then* $h_{2m+1} = 0$ *for* $1 \leqslant m \leqslant n-1$.

Proof.

(i) Suppose H_{i-1} is the jth non-vanishing member in list $\{H_1, \cdots, H_n\}$, then, $i-1 = q_j$, $i+1 = q_{j+1}$. Therefore,

$$s_j = q_{j+1} - q_j = 2.$$

From Corollary 1 (ii), we know that

$$\frac{H_{i+1}}{H_i} = (-1)^{(s_j-1)s_j/2}(\overline{r_j r_{j+1}})^{s_j} = -(\overline{r_j r_{j+1}})^2 < 0.$$

Similarly, we know that if

$$H_{i-1} = H_i = H_{i+1} = 0, \quad H_{i-2} \cdot H_{i+2} \neq 0,$$

then $H_{i-2} \cdot H_{i+2} > 0$.

(ii) Suppose $q_{k-1} < m < m+1 < q_k$. Do the same transformations to h_{2m+1} as what we did to A_m in the proof of Theorem 5, keeping the last row unchanged. We have

$$
h_{2m+1} \to
\left(\begin{array}{cccccccccc}
r_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
 & \ddots & & & & \ddots & & \ddots & & \\
 & & r_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & & 0 & \cdots & 0 & r_k & & & \\
 & \vdots & \vdots & & \ddots & & \ddots & \ddots & & \\
 & & & & & 0 & \cdots & 0 & r_k & \\
\cdots & a_0 & a_1 & \underbrace{a_2 & \cdots}_{n_1} & & \cdots & & &
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m - q_{k-2} \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} m - q_{k-1} + 1 \\
\end{array}
$$

$$\xrightarrow{0}\left(\begin{array}{ccccccccc} r_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & \ddots & & & & \ddots & & \ddots & \\ & & r_{k-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & 0 & \cdots & 0 & r_k & & \\ & \vdots & \vdots & & \ddots & & \ddots & \ddots & \\ & & & & & 0 & \cdots & 0 & r_k \\ 0 & \cdots & 0 & \underbrace{a'_0 \quad a'_1 \quad \cdots}_{n_1} & & & & \cdots & \end{array}\right) \begin{array}{l} \left.\right\} m - q_{k-2} \\ \\ \left.\right\} m - q_{k-1} + 1 \end{array},$$

where $n_1 = s_{k-1} - (m - q_{k-1})$. Now, because $h_{2m+2} = 0$, $s_{k-1} - ((m+1) - q_{k-1}) > 0$, and therefore, $n_1 = s_{k-1} - (m - q_{k-1}) > 1$. Then, $h_{2m+1} = 0$ is obvious.

Let us now consider the discrimination matrix, Discr(f), of a polynomial $f(x)$,

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n (a_0 \neq 0).$$

It is easy to see (let $g(x) = f'(x)$) that Theorem 5 and Proposition 1 both hold for Discr(f). Furthermore, we have

Proposition 2.

(i) *In list* $\{d_1, d_3, \cdots, d_{2n+1}\}$, *if* $d_{2i+1} = 0$ *and* $d_{2i-1} \cdot d_{2i+3} \neq 0$ *for some* i, $1 \leqslant i \leqslant n-1$, *then* $d_{2i-1} \cdot d_{2i+3} < 0$.

(ii) *If* $d_{2m-1} = d_{2m+1} = 0$, *then* $d_{2m} = 0$ *for* $1 \leqslant m \leqslant n$.

Theorem 6[1)]. *Let* $g(x) = xf'(x)$ *and* A *be the Sylvester matrix of* $f(x)$ *and* $g(x)$, *also let* $H_0 = 1$ *and* H_1, $H_2, \cdots$, H_n *be the even order principal minor of* A. *If the number of sign changes of the revised sign list of* $\{H_0, H_1, \cdots, H_n\}$ *is* v *and* $H_l \neq 0$, $H_m = 0$ $(m > l)$, *then*

$$l - 2v = f_{(0,\infty)} - f_{(-\infty,0)},$$

where $f_{(0,\infty)}$ *means the number of distinct positive roots of* $f(x)$ *and* $f_{(-\infty,0)}$ *the number of distinct negative roots of* $f(x)$.

Theorem 7. *If the number of sign changes of the revised sign list of*

$$\{d_1, d_3, \cdots, d_{2n+1}\}$$

is v *and the number of non-vanishing members of that list is* $l+1$, *i.e.* $d_{2l+1} \neq 0$, $d_{2m+1} = 0$ $(m > l)$, *then*

$$l - 2v = f_{(-\infty,0)} - f_{(0,\infty)}.$$

Proof. First of all, if $t_0, t_1, \cdots, t_n$ is a sequence of non-vanishing real numbers, then its number of sign changes equals

$$\sum_{i=0}^{n-1} \frac{1}{2}(1 - \mathrm{sign}(t_i t_{i+1})).$$

Let $[\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_n]$ be the revised sign list of $\{H_0, H_1, \cdots, H_n\}$. Suppose its number of sign changes is v_1 and $\varepsilon_{l_1} \neq 0$, $\varepsilon_m = 0$ $(m > l_1)$, then by Theorem 6, we get

$$l_1 - 2v_1 = f_{(0,\infty)} - f_{(-\infty,0)}.$$

Let $[\varepsilon'_0, \varepsilon'_1, \cdots, \varepsilon'_n]$ be the revised sign list of $\{d_1, d_3, \cdots, d_{2n+1}\}$, we need to prove

$$l - 2v = -(l_1 - 2v_1).$$

Supposing $l_1 = q_k$, we have

$$\begin{aligned} l_1 - 2v_1 &= l_1 - 2\sum_{i=0}^{l_1-1} \frac{1}{2}(1 - \mathrm{sign}(\varepsilon_i \varepsilon_{i+1})) = \sum_{i=0}^{l_1-1} \mathrm{sign}(\varepsilon_i \varepsilon_{i+1}) \\ &= \sum_{i=0}^{k-1}\Big(\sum_{j=0,\ s_i>1}^{s_i-2} \mathrm{sign}(\varepsilon_{q_i+j}\varepsilon_{q_i+j+1}) + \mathrm{sign}(\varepsilon_{q_i+s_i-1}\varepsilon_{q_{i+1}})\Big) \\ &= \sum_{i=0}^{k-1}\Big(\sum_{j=0,\ s_i>1}^{s_i-2} (-1)^{\frac{j(j+1)}{2}+\frac{(j+1)(j+2)}{2}} \cdot \mathrm{sign}(\sigma_i^2) + (-1)^{(s_i-1)s_i/2}.\ \mathrm{sign}(\sigma_i\sigma_{i+1})\Big) \\ &= \sum_{i=0}^{k-1}\Big(\sum_{j=0,\ s_i>1}^{s_i-2} (-1)^{j+1} + (-1)^{(s_i-1)s_i} \cdot \mathrm{sign}((\overline{r_i r_{i+1}})^{s_i} \cdot \sigma_i^2)\Big) \\ &= \sum_{i=0}^{k-1}\Big(\frac{1}{2}((-1)^{s_i-1} - 1) + \mathrm{sign}((\overline{r_i r_{i+1}})^{s_i})\Big) \\ &= \sum_{i=0,\ s_i \mathrm{odd}}^{k-1} \mathrm{sign}\ (\overline{r_i r_{i+1}}). \end{aligned}$$

By relations,

$$d_{2i+1} = (-1)^i \cdot a_0 \cdot H_i (0 \leqslant i \leqslant n),$$

we know $l = q_k$ and

$$\varepsilon'_{q_i} = (-1)^{q_i} \varepsilon_{q_i},\ 0 \leqslant i \leqslant k.$$

So similarly,

$$l - 2v = \sum_{i=0}^{k-1}\Big(\sum_{j=0,\ s_i>1}^{s_i-2} \mathrm{sign}(\varepsilon'_{q_i+j}\varepsilon'_{q_i+j+1}) + \mathrm{sign}(\varepsilon'_{q_i+s_i-1}\varepsilon'_{q_{i+1}})\Big).$$

For every i $(0 \leqslant i \leqslant k-1)$, if both q_i and s_i are odd, then $q_{i+1} = q_i + s_i$ is even, so

1) Chen, M., Generalization of Discrimination System for Polynomials and Its Applications (in Chinese), Ph. D dissertation, Sichuan Union University, 1998.

$$\sum_{j=0,\ s_i>1}^{s_i-2} \operatorname{sign}(\varepsilon'_{q_i+j}\,\varepsilon'_{q_i+j+1}) + \operatorname{sign}(\varepsilon'_{q_i+s_i-1}\varepsilon'_{q_{i+1}})$$

$$= \frac{1}{2}((-1)^{s_i-1} - 1) - \operatorname{sign}((\overline{r_i r_{i+1}})^{s_i})$$

$$= -\operatorname{sign}(\overline{r_i r_{i+1}}).$$

If q_i is odd and s_i is even, then $q_{i+1} = q_i + s_i$ is odd, so

$$\sum_{j=0,\ s_i>1}^{s_i-2} \operatorname{sign}(\varepsilon'_{q_i+j}\,\varepsilon'_{q_i+j+1}) + \operatorname{sign}(\varepsilon'_{q_i+s_i-1}\varepsilon'_{q_{i+1}})$$

$$= \frac{1}{2}((-1)^{s_i-1} - 1) + \operatorname{sign}((\overline{r_i r_{i+1}})^{s_i})$$

$$= 0.$$

The other two cases, in which q_i is even and s_i is odd or both q_i, and s_i are even can also be discussed as the same. Finally we get

$$l - 2v = -\sum_{i=0,\ s_i \text{ odd}}^{k-1} \operatorname{sign}(\overline{r_i r_{i+1}})$$

$$= -(l_1 - 2v_1).$$

Theorem 8. *Let* $\{d_1, d_2, \cdots, d_{2n}, d_{2n+1}\}$ *be the principal minor sequence of* $\mathrm{Discr}(f)$, *the discrimination matrix of a polynomial* $f(x)$,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n (a_0 \neq 0,\ a_n \neq 0).$$

And let l_1, v_1; l_2, v_2; l, v *be the numbers of non-vanishing members and sign changes of the revised sign lists of*

$$\{d_2, d_4, \cdots, d_{2n}\},$$

$$\{d_1, d_3, \cdots, d_{2n+1}\},$$

and

$$\{d_1d_2, d_2d_3, \cdots, d_{2n}d_{2n+1}\}$$

respectively. Then $l = l_1 + l_2 - 1$, $v = v_1 + v_2$.

Proof. By Theorem 4, we know the number of distinct negative real roots of $f(x)$ equals $l/2$

$-v$. On the other hand, by Theorems 1 and 7, we know this number also equals $(l_1 + l_2 - 1)/2 - (v_1 + v_2)$. So, if $l = l_1 + l_2 - 1$, then $v = v_1 + v_2$. By Propositions 1 (ii) and 2 (ii), we have $|2l_1 - (2l_2 - 1)| = 1$. Obviously, l must be even, so $l = 2l_1$, and $2l_1 < (2l_2 - 1)$; therefore,

$$2l_2 - 1 - 2l_1 = 1, \text{ i.e. } l_2 = l_1 + 1.$$

Finally, we get $l = 2l_1 = l_1 + l_2 - 1$.

4 Examples

Problem 1. *Find the number of positive roots of* $x^{11} - 3x^8 + 2x^3 - 5x + 6$.

This is equivalent to find the number of the negative roots of

$$x^{11} + 3x^8 + 2x^3 - 5x - 6.$$

The sign list of its n.r.d. is

$$[1,0,0,0,0,1,1,0,0,1,1,-1,-1,-1,1,-1,1,-1,-1,-1,-1,-1],$$

and the revised sign list is

$$[1,-1,-1,1,1,1,1,1,-1,-1,1,1,-1,-1,-1,1,-1,1,-1,-1,-1,-1,-1],$$

where the number of sign changes and the number of non-vanishing terms are 9 and 22, respectively. So, by Theorem 4, $f(x)$ has 2 distinct negative roots, hence the original polynomial has 2 distinct positive roots. No repeated roots exist in this case because $D_{11}(f) = d_{22} \neq 0$.

Problem 2. *Find the condition on* a, b, c *such that*

$$(\forall x > 0) x^5 + ax^2 + bx + c > 0.$$

This is equivalent to find the sufficient and necessary condition such that polynomial

$$f(x) = x^5 - ax^2 + bx - c$$

has no negative root(s). Using Theorems 1,4 and 8, by a thorough and detailed analysis on the sign list of the n.r.d. of $f(x)$, we conclude that

$$(\forall x > 0) x^5 + ax^2 + bx + c > 0$$

if and only if one of the following 6 cases applies

(1) $c > 0 \wedge d_{10} > 0 \wedge (a \geqslant 0 \vee d_9 \geqslant 0)$,

(2) $c > 0 \wedge d_{10} \leqslant 0 \wedge a \geqslant 0 \wedge d_9 < 0 \wedge d_8 < 0$,

(3) $c > 0 \wedge d_{10} = 0 \wedge d_8 = 0 \wedge a \geqslant 0$,

(4) $c = 0 \wedge b > 0 \wedge d'_8 > 0$,

(5) $c=0 \wedge b>0 \wedge d'_8 \leqslant 0 \wedge a>0$,

(6) $c=0 \wedge b=0 \wedge a \geqslant 0$,

where

$$d_8 = -27a^4 - 300abc + 160b^3,$$

$$d_9 = -27a^4 b + 225c^2 a^2 - 720cab^2 + 256b^4,$$

$$d_{10} = -27b^2 a^4 + 108a^5 c - 1600acb^3 + 2250a^2 bc^2 + 256b^5 + 3125c^4,$$

$$d'_8 = -27a^4 + 256b^3.$$

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